CSC373 Week 5 Notes

- Intro to Linear Programming (LP):
- With LP, we want to max or min an objective function subject to various constraints.
- $f: \mathbb{R}^n \to \mathbb{R}$ is a linear function if $f(x) = a^T x$ for some $a \in \mathbb{R}^n$.

E.g. $f(x_1, x_2) = 3x_1 - 5x_2$

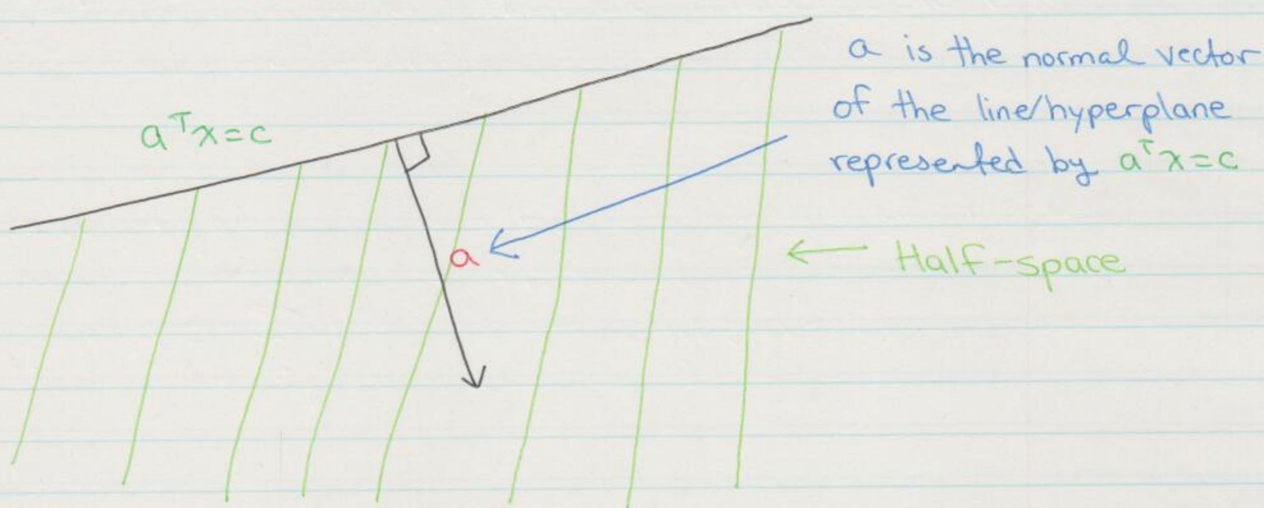
$$= \underbrace{\begin{pmatrix} 3 \\ 5 \end{pmatrix}^T}_{a} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}$$

- Our objective function and constraints are all linear functions.

- For constraints, we have $g(x) = c$ where $g: \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$.

- $a^T x \le c$ represents a half-space.

E.g.

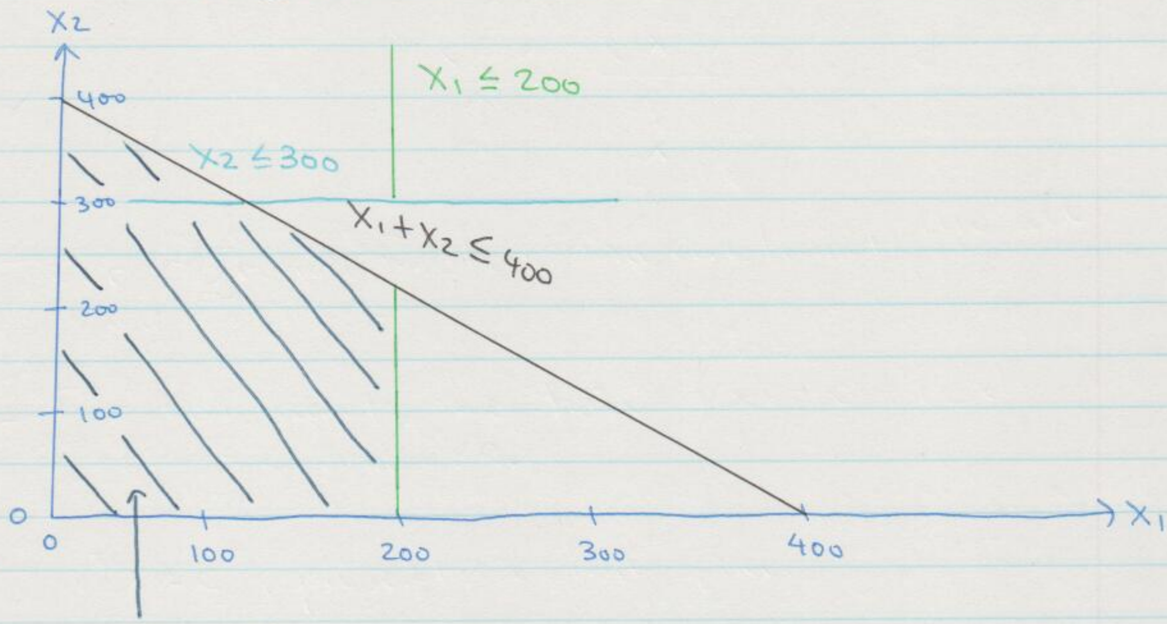


- E.g.

$$\max\ x_1 + 6x_2 \longleftarrow \text{Objective function}$$

$$\left.\begin{aligned} x_1 &\le 200 \\ x_2 &\le 300 \\ x_1 + x_2 &\le 400 \\ x_1, x_2 &\ge 0 \end{aligned}\right\} \text{Constraints}$$

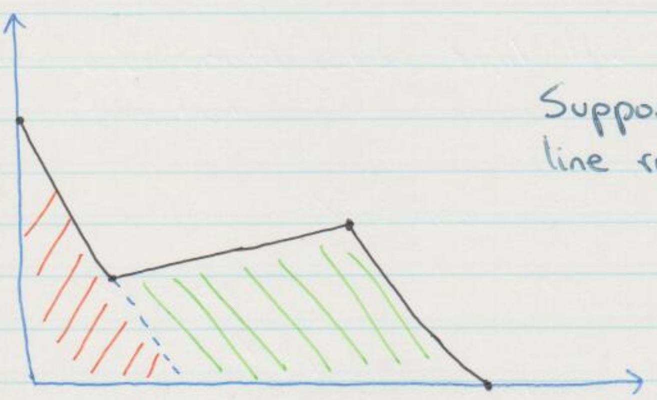
Note: We could also find the min of the objective function.



The feasible region.
This area is the intersection of the constraints.

- Convexity:
- Regardless of the obj func, there must be a vertex that is an optimal soln.
- This is because the feasible region must be convex.

Consider this concave feasible region:



Suppose the blue-dotted line represents a constraint.

This feasible regions allows areas on both sides of the dotted line to be part of it. (See the orange and green shaded areas). However, this is not allowed for LP. In LP, only 1 side of the constraint function can be part of the feasible region. Hence, all feasible regions must be convex.

- Furthermore, consider this "proof":

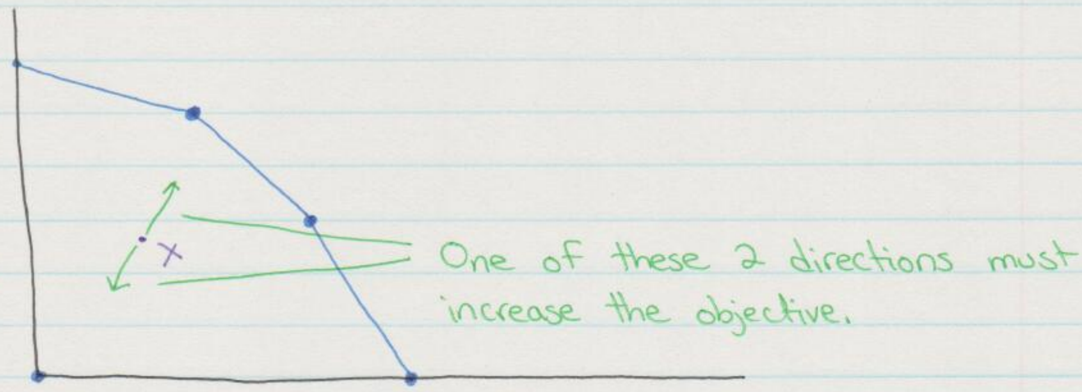
Start at some point, x, in the feasible region.
If x is not a vertex:

Find a direction d s.t. points within a positive distance of ε from x in both d and $-d$ are still in the feasible region.

The objective must not decrease for one of the 2 directions.

Repeat until we reach a vertex.

I.e.



- Standard Formulation:
- Let $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

- Let $a_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}, \quad 1 \le i \le m$

- Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

- Let $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

- The standard form says:
  1. Maximize $z = c^T x$
     $= c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

     Objective function → ; ↑ n variables

  2. Subject to the constraints:

$$a_1^T x \leq b_1 \rightarrow a_{11} x_1 + \ldots + a_{1n} x_n \leq b_1$$
$$a_2^T x \leq b_2 \rightarrow a_{21} x_1 + \ldots + a_{2n} x_n \leq b_2$$
m constraints →
$$\vdots$$
$$a_m^T x \leq b_m \rightarrow a_{m1} x_1 + \ldots + a_{mn} x_n \leq b_m$$

  and

  n more constraints → $x \geq 0$

  I.e. We want to max $c^T x$ subject to
  1. $Ax \leq b$
  2. $x \geq 0$

- If a constraint uses $\geq$, we can do:
  $a^T x \geq b \iff -a^T x \leq -b$ (Multiply both sides by -1)

- If a constraint uses equality, =, we can do:
  $a^T x = b \iff a^T x \leq b,\ a^T x \geq b$
  I.e. We can split = into $\leq$ and $\geq$
  Note: We can use the above method to change $\geq$ to $\leq$.
- If we're asked to min the objective function, we can max its negative:
  Min $c^T x \iff$ Max $-c^T x$

- If a var, x, is unconstrained, we can replace x by 2 variables x' and x'' s.t. we replace each occurrence of x with $x' - x''$ and set $x' \geq 0$, $x'' \geq 0$.

- E.g. Transform the LP problem below to Standard Form

Min $-2x_1 + 3x_2$

Subject to

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \leq 4$$
$$x_1 \geq 0$$

Soln:

Here are the things we have to change:

1. Min obj function
2. $x_1 + x_2 = 7$
3. x_2 has no constraint

I'll tackle #3 first.

Replace x_2 with $x_2' - x_2''$ s.t. $x_2' \geq 0$, $x_2'' \geq 0$

I'll tackle #2 now.

Replace $x_1 + x_2' - x_2'' = 7$ with

1. $x_1 + x_2' - x_2'' \leq 7$
2. $x_1 + x_2' - x_2'' \geq 7$

   $\hookrightarrow -x_1 - x_2' + x_2'' \leq 7$

I'll tackle #1 now.

Replace min $-2x_1 + 3x_2' - 3x_2''$ with

max $2x_1 - 3x_2' + 3x_2''$

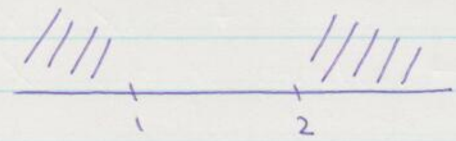
This is the Standard Form:

Max $2x_1 - 3x_2' + 3x_2''$

Sub To

$$x_1 + x_2' - x_2'' \le 7$$
$$-x_1 - x_2' + x_2'' \le -7$$
$$x_1 - 2x_2' + 2x_2'' \le 4$$
$$x_1, x_2', x_2'' \ge 0$$

- An LP doesn't always have an optimal solution. It can fail for 2 reasons:
  1. It is infeasible. I.e. $\{x \mid Ax \le b\} = \emptyset$
     E.g. The set of constraints is $\{x_1 \le 1, -x_1 \le -2\}$

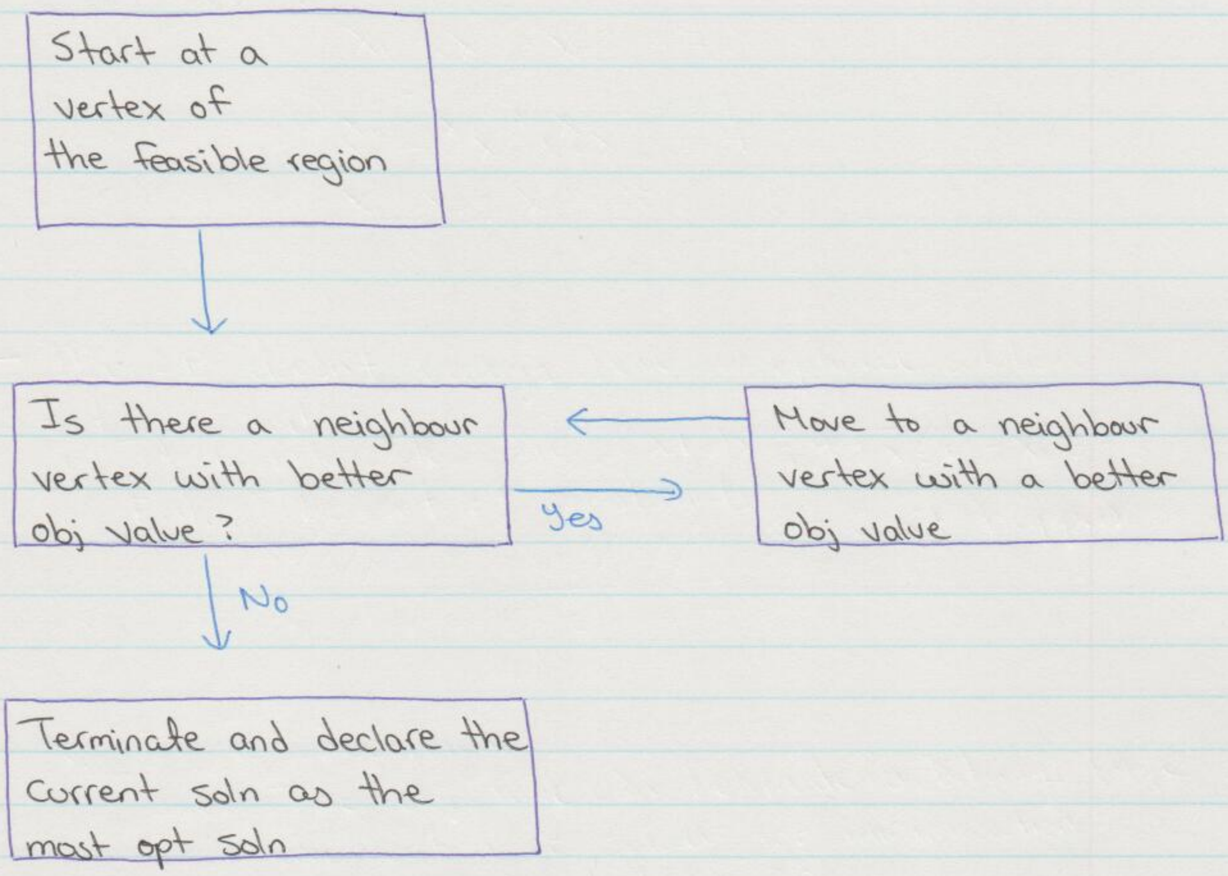


← Can't happen

  2. It is unbounded.
     I.e. The obj func can be made arbitrary large for maximization or small for minimization.
     E.g. Max x_1 subject to $x_1 \ge 0$

- Simplex Algorithm:
- Algorithm:
  let v be any vertex of the feasible region
  while there is a neighbour v' of v with better obj value:
      set v to v'

- I.e.



- To implement this, we'll need to work with the Slack form of LP.

Standard form

Max $c^T x$

sub to

$$Ax \le b$$
$$x \ge 0$$

$\longrightarrow$

Slack form

$$z = c^T x$$
$$s = b - Ax$$
$$x, s \ge 0$$

Note: $Ax \le b$

$\rightarrow 0 \le b - Ax$ s.t. $x \ge 0$

$\Leftrightarrow s = b - Ax$ s.t. $x \ge 0,\ s \ge 0$

- E.g.

Standard Form

$$\left\{\begin{array}{ll} \text{Max} & 2x_1 - 3x_2 + 3x_3 \\ \text{Subject To} & \\ & x_1 + x_2 - x_3 \le 7 \\ & -x_1 - x_2 + x_3 \le -7 \\ & x_1 - 2x_2 + 2x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{array}\right.$$

$$\Updownarrow$$

$$\text{Max} \quad \overbrace{2x_1 - 3x_2 + 3x_3}^{\text{Non-basic Var}}$$

Subject to

$$\text{Basic Var} \left\{\begin{array}{l} x_4 = 7 - x_1 - x_2 + x_3 \\ x_5 = -7 + x_1 + x_2 - x_3 \\ x_6 = 4 - x_1 + 2x_2 - 2x_3 \end{array}\right.$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$$

- E.g.

Step 1

Start at a feasible vertex.

For now, assume $b \ge 0$ (I.e. Each $b_i \ge 0$)

In this case, $x=0$ is a feasible vertex

In slack form, this means setting the non-basic vars to 0.

$$\left.\begin{array}{l} z = 3x_1 + x_2 + 2x_3 \\ x_4 = 30 - x_1 - x_2 - 3x_3 \\ x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \\ x_6 = 36 - 4x_1 - x_2 - 2x_3 \\ x_1, x_2, x_3, x_4, x_5, x_6 \ge 0 \end{array}\right\} \text{Starting point}$$

Step 2

To increase the value of z, find a non-basic var with a positive coefficient (this is called an entering var) and see how much we can increase its value without violating any constraints.

From the slack form on the prev page, I'll increase x_1.

$$x_4 = 30 - x_1 - x_2 - 3x_3 \rightarrow x_4 = 30 - x_1$$
$$x_1 = 30 - x_4$$
$$\leq 30$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \rightarrow x_5 = 24 - 2x_1$$
$$2x_1 = 24 - x_5$$
$$\leq 24$$
$$x_1 \leq 12$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$
$$\rightarrow x_6 = 36 - 4x_1$$
$$4x_1 = 36 - x_6$$
$$\leq 36$$

Tightest bound $\rightarrow$ $x_1 \leq 9$

Note: x_2 and x_3 are 0 from step 1 and we know $x_4, x_5, x_6 \geq 0$

Now, we'll solve the tighest bound for the non-basic var.

$$x_1 = 9 - \frac{x_6}{4} - \frac{x_2}{4} - \frac{x_3}{2}$$

Now, we'll substitute the entering var (called pivot) in other eqns.

Now, x_1 is basic and x_6 is non-basic.

x_6 is called the leaving var.

I'll replace all instances of x_1 with $9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}
\end{aligned}
$$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

After 1 iteration of this step, the basic feasible soln (I.e. substituting 0 for all non-basic var) improves from $z=0$ to $z=27$.

We keep repeating this process until there is no entering var (I.e. There is no non-basic var with a positive coefficient)

I'll choose x_3 now. First, I'll find the tightest bound.

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \longrightarrow x_1 = 9 - \frac{x_3}{2}$$

$$
\begin{aligned}
2x_1 &= 18 - x_3 \\
x_3 &= 18 - 2x_1 \\
&\leq 18
\end{aligned}
$$

$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \longrightarrow x_4 = 21 - \frac{5x_3}{2}$$

$$
\begin{aligned}
\frac{2x_4}{5} &= \frac{42}{5} - x_3 \\
x_3 &= \frac{42}{5} - \frac{2x_4}{5} \\
&\leq \frac{42}{5}
\end{aligned}
$$

$$x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} \longrightarrow x_5 = 6 - 4x_3$$

$$
\begin{aligned}
\frac{x_5}{4} &= \frac{3}{2} - x_3 \\
x_3 &= \frac{3}{2} - \frac{x_5}{4} \\
&\leq \frac{3}{2}
\end{aligned}
$$

Tightest bound $\longrightarrow$ $\leq \frac{3}{2}$

Now that we've found our tightest bound, we'll solve it for our non-basic var (x_3) and then substitute it in other equations.

$$4x_3 = 6 - \frac{3x_2}{2} - x_5 + \frac{x_6}{2}$$
$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$

$$Z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16}$$
$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16}$$
$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$
$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Now, I'll choose x_2 as the entering var.

$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \longrightarrow x_1 = \frac{33}{4} - \frac{x_2}{16}$$
$$x_2 = 132 - 16x_1$$
$$\leq 132$$

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \longrightarrow x_3 = \frac{3}{2} - \frac{3x_2}{8}$$
$$x_2 = 4 - \frac{8}{3}x_3$$
Tightest Bound $\longrightarrow$ ≤ 4

$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} \longrightarrow x_4 = \frac{69}{4} + \frac{3x_2}{16}$$
$$\frac{16x_4}{3} = 92 + x_2$$
$$\frac{16x_4}{3} - 92 = x_2$$

Now that we've found our tightest bound, we'll solve it for our non-basic var (x_2) and then substitute it in other eqns.

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$
$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}$$

$$z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3}$$
$$x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3}$$
$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}$$
$$x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2}$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Since there are no more entering vars (Non-basic vars with a positive coefficient), we're done.

Take the basic feasible soln ($x_3 = x_5 = x_6 = 0$) and that gives us $\underbrace{z = 28.}_{\text{Optimal value of } z}$

In the opt soln, $x_1 = 8$
$x_2 = 4$
$x_3 = 0$

- Some Outstanding Issues:

1. What if the entering var has no upper bound? If it doesn't appear in any constraints or only appears in constraints where it can go to infinity, then z can also go to ∞, so declare that LP is unbounded.

2. What if pivoting doesn't change the constant in z?
This is known as degeneracy and can lead to infinite loops.
This can be prevented by perturbing b by a small random amount in each coordinate or by carefully breaking ties among entering and leaving vars.
(Bland's Rule)

3. Earlier, we assumed $b \geq 0$ and started with the vertex $x=0$. What if this doesn't hold?

LP1

Max $c^T x$

s.t.

$$\begin{aligned} a_1^T x &= b_1 \\ a_2^T x &= b_2 \\ &\vdots \\ a_m^T x &= b_m \\ x &\geq 0 \end{aligned}$$

$\longrightarrow$

LP2

Max $c^T x$

s.t.

$$\begin{aligned} a_1^T x + s_1 &= b_1 \\ a_2^T x + s_2 &= b_2 \\ &\vdots \\ a_m^T x + s_m &= b_m \\ x, s &\geq 0 \end{aligned}$$

LP3

Max $c^T x$

s.t.

Here, since $b \leq 0$, we multiply each side by -1 to make the RHS positive.

$$\left.\begin{aligned} -a_1^T x - s_1 &= -b_1 \\ -a_2^T x - s_2 &= -b_2 \\ &\vdots \\ -a_m^T x - s_m &= -b_m \end{aligned}\right\}$$

$$x, s \geq 0$$

$\longrightarrow$

LP4

Min $\sum_i z_i$

s.t.

$$\begin{aligned} -a_1^T x - s_1 + z_1 &= -b_1 \\ -a_2^T x - s_2 + z_2 &= -b_2 \\ &\vdots \\ -a_m^T x - s_m + z_m &= -b_m \\ x, s, z &\geq 0 \end{aligned}$$

In LP4, the minimal value of $\sum_i z_i$ is 0, since $z \geq 0$. If $z = 0$, we get an x^* and s^* s.t. $Ax + s = b$.

I.e. When $z = 0$, and we get an x^* and s^*, it is the basic, feasible soln of LP1.

So now:

1. We solve LP4 using the simplex algo with the initial basic soln being $x = s = 0$ and $z = |b|$.
2. If its optimum value is 0, extract a basic feasible soln x^* from it and use it to solve LP1 using the simplex algo.

   If its opt value is greater than 0, then LP1 is infeasible.

- Dual LP:
- Suppose we have the following LP problem in standard form:

Max $\quad x_1 + 2x_2 + x_3 + x_4$

Subject to

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &\leq 2 \\
x_2 + x_4 &\leq 1 \\
x_1 + 2x_3 &\leq 1 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

and some LP-solver tells us that the soln is $x_1 = 1$, $x_2 = \frac{1}{2}$, $x_3 = 0$, $x_4 = \frac{1}{2}$, cost $= 2.5$. How can we verify it without retracing our steps?

Suppose if we have 2 inequalities

1. $a \le b$
2. $c \le d$

Then:

1. $m \cdot a \le m \cdot b$, for all $m \ge 0$
2. $\underbrace{a+c}_{\text{LHS}} \le \underbrace{b+d}_{\text{RHS}}$
3. $m_1 \cdot a + m_2 \cdot c \le m_1 \cdot b + m_2 \cdot d$

Going back to our LP problem, if we scale the first inequality by $\frac{1}{2}$, and scale the third inequality by $\frac{1}{2}$ and add all 3 inequalities, we get

$$x_1 + 2x_2 + \frac{3}{2}x_3 + x_4 \le 2.5$$

Hence, for every feasible (x_1, x_2, x_3, x_4) its cost is ≤ 2.5.

Now, we want to find a way to get good scaling factors.

Suppose we have this LP problem in Standard Form:

Max $\quad c_1x_1 + \dots + c_nx_n$

Subject to

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n &\le b_m \\ x_1, \dots, x_n &\ge 0 \end{aligned}$$

Let's focus on the $Ax \leq b$ part for now.
Let $y_1, \ldots, y_m \geq 0$ be some scaling factors.
Then, we can get:

$$y_1(a_{11}x_1 + \ldots + a_{1n}x_n) \leq b_1 \cdot y_1$$
$$y_2(a_{21}x_1 + \ldots + a_{2n}x_n) \leq b_2 \cdot y_2$$
$$\vdots$$
$$y_m(a_{m1}x_1 + \ldots + a_{mn}x_n) \leq b_m \cdot y_m$$

Now, if we add up the inequalities, we get:

$$y_1(a_{11}x_1 + \ldots + a_{1n}x_n) + \ldots + y_m(a_{m1}x_1 + \ldots + a_{mn}x_n) \leq b_1 y_1 + \ldots + b_m y_m$$

Rearranging the above inequality, we get:

$$(a_{11}y_1 + \ldots + a_{m1}y_m)x_1 + \ldots + (a_{1n}y_1 + \ldots + a_{mn} \cdot y_m) \cdot x_n \leq b_1 y_1 + \ldots + b_m y_m$$

Now, the trick is to choose the y_i s.t. the linear function of the x_i is an upper bound to the cost function.
I.e.

$$c_1 \leq a_{11}y_1 + \ldots + a_{m1}y_m$$
$$c_2 \leq a_{12}y_1 + \ldots + a_{m2}y_m$$
$$\vdots$$
$$c_n \leq a_{1n}y_1 + \ldots + a_{mn}y_m$$

Multiply both sides by x_i and adding the inequalities, we get:

$$c_1x_1 + \ldots + c_nx_n \leq (a_{11}y_1 + \ldots + a_{m1}y_m)x_1 + \ldots + (a_{1n}y_1 + \ldots + a_{mn}y_m)x_n$$
$$\leq y_1b_1 + \ldots + y_mb_m$$

We want to find non-negatives values $y_1, \dots, y_m$ s.t. the upper bound is as tight as possible.

I.e. We want to do this:

Min $b_1 y_1 + \dots + b_m y_m$

Subject to

$$a_{11} y_1 + \dots + a_{m1} y_m \geq c_1$$
$$\vdots$$
$$a_{1n} y_1 + \dots + a_{mn} y_m \geq c_n$$
$$y_1, \dots, y_m \geq 0$$

So, if we want to find the scaling factors that give us the best possible upper bound to the opt soln of a LP in standard form, we end up with a new LP problem.

I.e.

If Max $c^T x$

Subject to

$$Ax \leq b$$
$$x \geq 0$$

Called Primal LP

is an LP in standard form, then its dual is

Min $b^T y$

Subject to

$$A^T y \geq c$$
$$y \geq 0$$

Called Dual LP

The dual is formed by:

1. Having one var for each constraint of the primal, not counting the non-negativity constraints.

2. Having one constraint for each var of the primal, plus the non-negativity constraints.

- Weak Duality:

Thm: For any primal feasible x and dual feasible y, $c^T x \le y^T b$

Proof: $c^T x \le (y^T A) x$
$= (y^T) A x$
$\le y^T b$